Spectral analysis of signals

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Level of study	PhD
Year of study	1
Semester	1
Duration	20 Hrs
ECTS	4
Course objective	Achieving the learning outcomes indicated below, notably deepening the student's knowledge and skills in the signal analysis and signal processing, along with the capability to apply the knowledge to practical engineering problems.
Learning outcomes	KW-1, KU-1, KK-1
Content	LECTURE (10 hours) Elementary continuous-time signals. Sampling and discrete-time signals. Frequency of discrete-time signals. Complex exponential signals and their orthogonality. Complex Fourier series of continuous-time and discrete-time signals. Fourier transform of continuous-time and discrete-time signals. Discrete Fourier transform (DFT). Recursive computation of DFT. Spectral factoring of PAM signals. Reconstruction of analog signals. Upsampling and downsampling. Spectral analysis of distorted power line waveforms (single-phase and three-phase); evaluation of harmonic distortion of voltages and currents. Related topics: phase locked-loop, coordinate transformations in multiphase systems, active power filters, spectrum analyzers for EMC applications etc. LABORATORY (10 hours) Using sampling and DFT for the analysis of selected continuous-time signals. Recursive implementation of DTF. Measurement of the fundamental harmonic of a distorted and/or amplitude-modulated signal. Implementation of the phase-locked loop (PLL). Example spectral analyses of distorted power line waveforms (single-phase and three-phase). Computing the total harmonic distortion (THD) of power line waveforms.
Suggested reading	1. Chen Chi-Tsong: *System and Signal Analysis*. 2nd edition, Saunders College Publishing, 1994 2. Jackson L.B.: *Digital Filters and Signal Processing*. 3rd edition, Kluwer Academic Publishers, 1996. 3. Lyons R.G.: *Understanding Digital Signal Processing*. 3rd Edition, Prentice Hall, 2010 (Polish translation available: *Wprowadzenie do cyfrowego przetwarzania sygnałów*. WKŁ 2010).
Grading method	• Test of lecture-related knowledge (50%) • Evaluation of laboratory assignments (50%)
Course language	English

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Example test questions

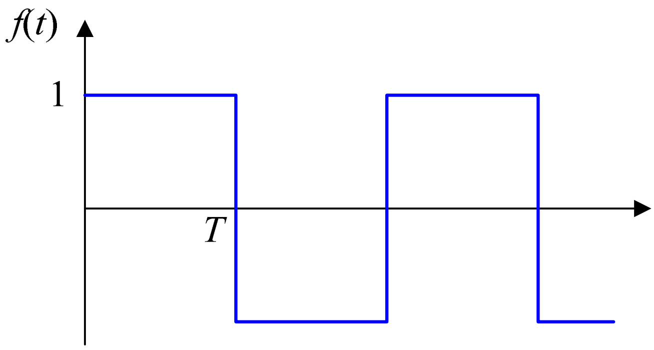
1. A periodic sequence of period N is made of the following samples (per period): 4, 2, 0, 3, 0, -3, 2, 0. Find the Fourier series coefficient c_2.

2. Find an approximate value of Fourier series coefficient c_{11}^{d} of a sequence obtained by sampling an analog periodic signal whose Fourier coefficients are given by

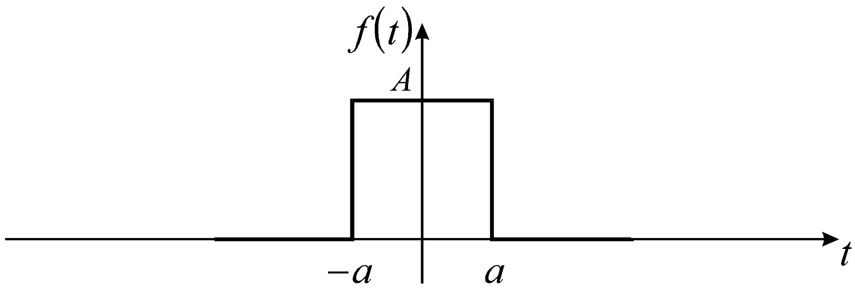
$$c_m^a = \frac{2m}{m^2+1}\ ,\ m \in \mathbf{Z}$$

The sampling frequency is 27 samples per period. The approximation should take into account 5 aliased harmonics of the analog signal.

3. Determine the Fourier series expansion of the analog periodic signal shown below using the spectral factorization into the discrimination factor and shape factor.



Hint: The Fourier transform of the rectangular pulse sketched below



is given by

$$F(\Omega) = 2aA \cdot \mathrm{sinc}\left(\frac{\Omega a}{\pi}\right)$$

Use this formula and the time-shifting property of the Fourier transform to find the required shape factor.

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Example laboratory tasks

Task 1 (waveform approximation by partial Fourier series)
Write a MATLAB function M-file with the following call syntax:

```
[y,t]=rectappr0(M)
```

computing an approximation of a square wave of amplitude $V_{\max}$ as a partial sum of its Fourier series. The approximation should be made of M odd real harmonics. The Fourier coefficients of the complex expansion can be evaluated using the following formula:

$$c_m = (-1)^{\frac{m-1}{2}} \cdot 2V_{\max} \cdot \frac{1}{m\pi}$$

The outputs of the function are as follows:

- `t` is a time vector that spans two fundamental periods and has sufficient resolution to ensure that the graph of the approximation created by the `plot` function is not visibly distorted;
- `y` is a vector of values of the approximating waveform at time points in `t`.

Useful relationships:

$$A_m \cos(m\Omega_0 t + \phi_m) = c_m e^{jm\Omega_0 t} + c_{-m} e^{-jm\Omega_0 t}$$

$$c_m = \frac{A_m}{2} \cdot e^{j\phi_m}, \quad c_{-m} = \frac{A_m}{2} \cdot e^{-j\phi_m}$$

where c_m denotes Fourier coefficients of the complex expansion.

Suggested tests:

Call the `rectappr0` function from MATLAB command window, starting with M=1

```
>> [y,t] = rectappr0(1);
```

and plot the result using

```
>> plot(t,y); grid
```

Repeat the above commands for increasing values of M and observe how the the approximation converges to the square wave. Make sure that the time resolution remains adequate as M increases. Note the appearance of the approximation around the points of discontinuity of the square wave.

Task 4 (recursive computation of DFT)
Write a MATLAB function M-file with the following call syntax:

```
c=recursive_dft(N,n,x)
```

implementing recursive computation of the Fourier series coefficients of discrete-time sequence `x`. The argument `n` is the order of the harmonic for which the coefficients are to be computed, while `N` denotes the period of the analyzed sequence (sequence `x` may contain transient states, but it is assumed that it is periodic with period `N` at steady state). The output `c` is a vector (time sequence) of Fourier series coefficients of the `n`-th harmonic of `x`.

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Hints:

Consider the conventional discrete Fourier transform (DFT):

$$X[m] = \sum_{k=0}^{N-1} x[k] \cdot e^{-jm\frac{2\pi}{N}k} \tag{1}$$

It computes the samples of the Fourier transform of sequence $x[k]$ based on N time samples at $k = 0, 1, \ldots, N-1$. The sequence can be considered finite-duration (with zero values at other sample points k) or periodic with period N. In the latter case, the DFT divided by N represents the coefficients of the Fourier series:

$$c_m = \frac{X[m]}{N} \tag{2}$$

Strictly speaking, a periodic signal never exhibits any changes in its repeatable pattern (the same periodic pattern is repeated over time from minus to plus infinity). As a practical matter, however, signals that are nominally periodic (such as the power line voltages and currents) inevitably exhibit various forms of sporadic or slow variation. Despite the fact that such signals are not *sensu stricto* periodic, it is sensible (and even necessary!) to evaluate their fundamental amplitude or rms value, as well as their harmonic content (both changing over time). To this end, we can evaluate the Fourier coefficients of interest over mowing time windows – in such a way that at a current sample time k the windows spans over the most recent N sample times (that is, k-N+1 through k). The moving-window DFT formula could be as follows:

$$X_m[k] \triangleq \sum_{r=k-N+1}^{k} x[r] \cdot e^{-jm\frac{2\pi}{N}r} = \sum_{r=k-N+1}^{k} x[r] \cdot W_m[r], \tag{3}$$

where $W_m = e^{-jm\frac{2\pi}{N}}$. In this case the Fourier transform samples are themselves time sequences (functions of discrete time k), which is reflected by the modified notation (the harmonic order m is now used as a subscript and the dependence of X_m on time k is explicitly indicated).

Consider the sets of values assumed by the bound variable r for $X_m[k-1]$ and $X_m[k]$.

$X_m[k-1]$:	k-N,	k-N+1, k-N+2, … , k-1	
$X_m[k]$:		k-N+1, k-N+2, … , k-1,	k

The comparison of the above sets suggests a simple method of recursive computation of $X_m[k]$ based on $X_m[k-1]$:

$$X_m[k] = X_m[k-1] - x[k-N] \cdot W_m[k-N] + x[k] \cdot W_m[k]$$
$$= X_m[k-1] - P[k-N] + P[k] \tag{4}$$

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When computing $X_m[k]$, the values of $X_m[k-1]$ and $P[k-N]$ should be available as previously computed and stored values. The evaluation of $X_m[k]$ only takes computing the new sample product $P[k]$, adding it to the previous transform value $X_m[k-1]$ and subtracting the oldest sample product $P[k-N]$ included in the computation of $X_m[k-1]$. Each newly computed sample product $P[k]$ effectively becomes $P[k-N]$ after N sample periods, meaning that it is necessary not only in the current computation, but will also be necessary in a future computation, and thus it should be stored over the consecutive N sample periods.

A convenient way to achieve this is to use a cyclic buffer of length N. The idea is illustrated in the figure below. Assume that at a given time k the circular buffer made of N cells stores N most recent sample products P[.] – denoted P[oldest_time] through P[oldest_time+N-1]. The latter product was computed at time k-1. At time k we will need the value of P[oldest_time] stored in the buffer, so the buffer address at time k (denoted cirbuf_address in the figure) should indicate the cell holding this value. Once P[oldest_time] has been retrieved and used in computing the new transform value, the cell can be used to store the new sample product for future use, whereupon the buffer address should be incremented to indicate the stored product value to be used in the next computation.

In constructing a MATLAB implementation of recursive DFT, use the following variables:

`Xm` – a scalar intended to store the current value of the transform (it corresponds to both $X_m[k]$ and $X_m[k-1]$ in eqn. 4);

`c` – a vector intended to store the time sequence of Fourier series coefficients of the analyzed sequence `x`; the entries in c are the consecutive values of `Xm/N`; the length of `c` is `length(x)`;

`cirbuff` – a vector of length N serving as a circular buffer;

`cirbuff_address` – a scalar used to address (index) the `cirbuff` (the address should be incremented modulo N to ensure that the buffer is circular); for the appropriate value of the address, `cirbuff(cirbuff_address)` will correspond to $P[k-N]$ in eqn. 4);

`new_product` – a scalar representing the product of the new signal sample and the appropriate complex exponential sample; it corresponds to $P[k]$ in eqn. 4.

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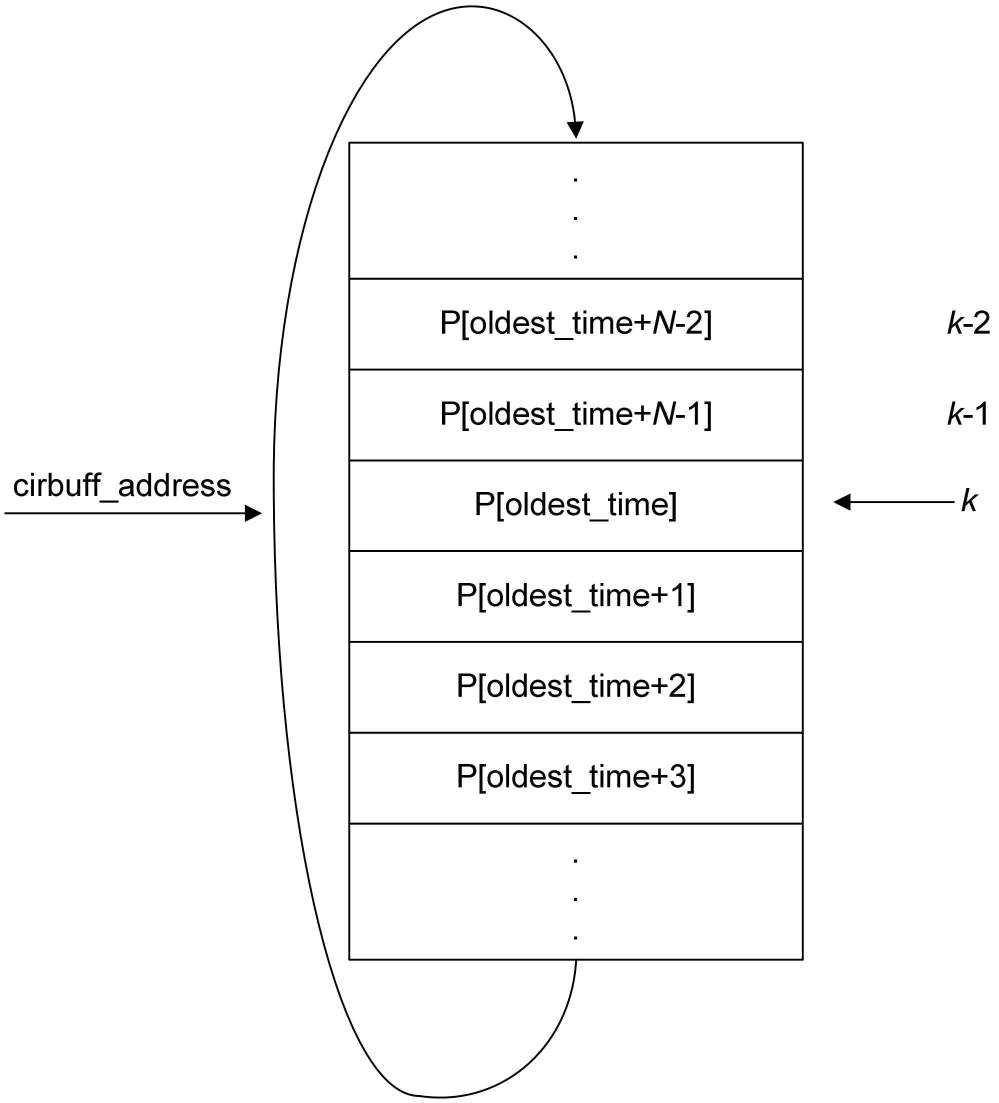

Using the above ideas, the computation of Xm might be performed in the following steps:

1) compute the `new_product` (that is, the product of the new signal sample and the appropriate complex exponential sample); new_product corresponds to $X_m[k]$ in (4);
2) compute `Xm = Xm - cirbuff(cirbuff_address)+ new_product` (that is, compute eqn. 4);
3) store `new_product` in `cirbuff(cirbuff_address)` for future use;
4) increment `cirbuff_address` (with reduction modulo *N*);
5) append `Xm/N` to `c` (i.e. the vector of Fourier series coefficients);
6) go back to step 1 (until the last sample in x has been reached).

Suggested tests:

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1) Synthesize a sequence made of a fundamental frequency sine wave of unity amplitude and a third harmonic of amplitude 0.3. The sequence should span two fundamental periods. Analyze the sequence using `recursive_dft`. Start by computing the Fourier coefficients corresponding to the fundamental harmonic, then plot and interpret the coefficient sequence. Repeat the same for the third harmonic.
2) Synthesize a sequence made of a sine wave which is amplitude-modulated by a piecewise-constant modulating sequence. The initial section of the modulating sequence should have amplitude 1.0 and the subsequent sections each should have amplitude increased by 0.1 compared to the preceding section. The length of each section should be equal to 10 periods of the sine wave. Both sequences should span 100 periods of the sine wave. Analyze the sequence using `recursive_dft`. Plot and interpret the coefficient sequence.

TERMINY WYKLADÓW			
Data	**Dzień tygodnia**	**Godzina**	**Sala**
2014-04-02	środa	13.15-15.15	WEiA E28
2014-04-09	środa	13.15-15.15	WEiA E28
2014-04-16	środa	13.15-15.15	WEiA E28
2014-04-23	środa	13.15-15.15	WEiA E28
2014-05-07	środa	13.15-15.15	WEiA E28
2014-05-14	środa	13.15-15.15	WEiA E28
2014-05-21	środa	13.15-15.15	WEiA E28
2014-05-28	środa	13.15-14.00	WEiA E28

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